VaR Computation of Non-Gaussian Stochastic Model

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***Abstract*—This paper proposes the *VaR* computation of non-Gaussian stochastic model. The problem of the *VaR* evaluation comes from the fact that it is not easy to estimate volatility. We present non-Gaussian models using stochastic volatility model where volatility is governed by logarithmic Ornstein-Uhlenbeck process. First, we show how to estimate the volatility with The Kalman filter procedure, and second we show how to extend the classical VaR techniques in an operational way using the stochastic process taking into account the asymmetric and leptokurtic distributions.**

***Index Terms*—VaR, stochastic volatility, Kalman filter, Ornstein-Uhlenbeck process, non-Gaussian distributions.**

I. Introduction

Despite a series of reforms, Basel II showed quickly its limits with the 2007 crisis, severely impacted the financial markets and the world economy in general. In this context, the Basel Committee has developed in the Basel III framework, a number of requirements that are intended to strengthen the resilience of banks and the financial system. The crisis of 2007 revealed an underestimation of risk measured by *VaR* models during periods of high volatility, because in Basel I and II, the *VaR* indicator was built under the assumption that the considered risk has a normal distribution. The worry of investors on volatility is omnipresent. The stake posed by the erratic nature of volatility led to the search of different *VaR* models, differing mainly in the nature of the process that implementing what type of process could determine a formula for the valuation performing the best possible compromise between practicality and reality?

Several studies have developed the *VaR* model. These include [1], [2] and [3] who were interested in the *VaR* evaluation in a static framework. However, [4] and [5] urge the study of models in a dynamic environment, but their studies are questionable as the volatility is assumed constant. However, in a near perspective of reality, volatility is defined by a random process. Thus, the empirical or economic result of the *VaR* computation is based on the quality of the estimate of volatility.

One of the main phenomena observed in financial markets is the presence of strong and sudden variations, often concentrated in time, forming turbulent times. Statistically, the existence of these extraordinary movements resulting in fat tails in the distribution function, calling into question the very strong assumption of Gaussian finance.

Many efforts were made to develop financial models arriving to take into account the statistical properties of market fluctuations. The finance literature has focused principally on two approaches: the autoregressive models *ARCH* (Auto Regressive Conditional heteroskedasticity) introduced by [6] and the non-deterministic modeling of volatility based on the work of [7]. However, the objection that may make against *ARCH* models is their deterministic scheme, while the other approach leads to volatility of a random variable.

Our aim is to provide a method for estimating the parameters of a stochastic volatility model and to present a procedure for evaluating the *VaR* of non-Gaussian stochastic model.

This article is organized as follows: Section II is devoted to the estimation method for stochastic volatility using the Kalman Filter. Section III determines the estimation procedure for *VaR*. Finally, there's a conclusion to end the article.

II. Estimation Method for Stochastic Volatility Using the Kalman Filter

In this study, the logarithm of the instantaneous volatility follows the process of mean-reverting, also called the process "*Ornstein-Uhlenbeck*". These reflect the presence of a force reverting to a long-term drift of the volatility. In particular, if *S* represents the observed value and if $\ln\sigma_t^2$ symbolizes the logarithm of the instantaneous volatility, the dynamics of the bivariate diffusion process are governed by the following equations:

$$\begin{cases} dS_t = \alpha_c S_t dt + \sigma_t S_t dB_t \\ d\ln\sigma_t^2 = \kappa\left[\vartheta - \ln\sigma_t^2\right]dt + \gamma dW_t \end{cases} \tag{1}$$

Brownian movement dB_t and dW_t are possibly correlated, $E(dB_t\ dW_t = \rho dt$ .The coefficient ρ has a strong economic content because it represents the leverage effect. The parameters to estimate, for this type of model is the drift α_c, the long-term average ϑ, the

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speed of mean reversion κ and volatility of volatility γ. These four parameters are considered constant, they will be determined from the database of the rate between time t_0 and T, which are respectively the first and last date in the database.

To estimate model (1), [8] conducted a linearization. This transformation helps to develop a model state measure. However, it should develop the *log-OU* by discretizing with a step equal to that of observations, denote by Δ the step of the discretization, for daily data, $\Delta = 1/252$ where approximately *252* is being the number of working days in a year, the discretization *Euler* gives:

$$\begin{cases} S_{t+\Delta} - S_t = \alpha_c S_t \Delta + \sigma_t \sqrt{\Delta} S_t u_t \\ \ln \sigma_{t+\Delta}^2 - \ln \sigma_t^2 = \kappa\left[\vartheta - \ln \sigma_t^2\right]\Delta + \gamma\sqrt{\Delta} v_t \end{cases} \quad (2)$$

u_t and v_t, two error terms are Gaussian with zero mean and variance 1. This model can be rewritten as follows:

$$\begin{cases} Y_t = \alpha + (\exp \frac{h_t}{2})\sqrt{\Delta} u_t \\ h_{t+\Delta} = \mu + \beta h_t + \delta v_t \end{cases} \quad (3)$$

Y_t indicates the return of the support, $h_t = \ln \sigma_t^2$, $\alpha = \alpha_c \Delta$, $\mu = \kappa\, \vartheta\, \Delta$, $\beta = 1 - \kappa\, \Delta$ and $\delta = \gamma\sqrt{\Delta}$.

Finally, the variations between two observations are modeled following the standard form:

$$\begin{cases} Y_t = \alpha + (\exp \frac{h_t}{2}) u_t \\ h_t = \mu + \beta h_{t-1} + \delta v_t \end{cases} \quad (4)$$

Formally adopting the notation $Y = (Y_1 ... Y_T)$ of the return vector of Support, $h = (h_1 ... h_T)$ unobservable volatility vector and denote by θ the set of parameters $(\alpha, \mu, \beta, \delta)$, the parameter α must be estimated prior to effecting the linearization of the model. The estimator of α is:

$$\hat{\alpha} = \frac{1}{T}\sum_{t=1}^{T} Y_t \quad (5)$$

To linearize the first equation of system (4), we square $Y_t - \hat{\alpha}$ and express it in logarithmic form. The following is obtained:

$$\begin{cases} \ln\left((Y_t - \hat{\alpha})^2\right) = h_t + \ln\left(u_t^2\right) \\ h_t = \mu + \beta h_{t-1} + \delta v_t \end{cases} \quad (6)$$

As further developing this equation as we know that $u_t \sim N\ (0.1)$, we can therefore deduce the distribution of $\ln\left(u_t^2\right)$. It corresponds to a logarithmic χ^2 distribution, whose expectation is -1.27 and the variance is $0.5\,\pi^2$, approximately 4.93. Note however that $\ln\left(u_t^2\right)$ cannot be correctly approximated by a normal law only if the sample is very large.

Following the approach of [9], by adding and subtracting $E\ln\left(u_t^2\right)$ in the first equation of model (6), we obtain:

$$\ln\left((Y_t - \hat{\alpha})^2\right) = E\ln\left(u_t^2\right) + h_t + \left[\ln\left(u_t^2\right) - E\ln\left(u_t^2\right)\right]$$

Setting $\xi = \left[\ln\left(u_t^2\right) - E\ln\left(u_t^2\right)\right]$ and $\zeta = \delta\, v_t$, we get two white noise centered on variance 4.93 and δ^2 respectively.

We can therefore rewrite the model (6) as follows:

$$\begin{cases} Y_t^* = c - 1.27 + h_t + \xi_t \\ h_t = \mu + \beta\, h_{t-1} + \zeta \end{cases} \quad (7)$$

where

$Y_t^* = \ln\left((Y_t - \hat{\alpha})^2\right)$ and *c* is a constant introduced to take into account that it equals only -1.27 in very large samples. This model is as state-space linear. Equations 1 and 3 of the model (7) are in the appropriate form to use the Kalman filter. The first equation is called measurement equation as the variable Y is observed. The second equation is the equation of state or transition as h the state variable is latent. The method of estimating this model is then explained in two steps: first, the latent variables are estimated by Kalman filter approach, and then the parameters are estimated by the method of Maximum Likelihood.

III. VaR Value of Non-Gaussian Model

By definition, the risk indicator called value at risk or simply *VaR* measures the worst expected loss over a given horizon at a given confidence level. Mathematically defined by:

$$\Pr\left(\Delta X(t) \le -VaR(t)\right) = \pi \quad (8)$$

The centering and the reduction of this relation give us:

$$\Pr\left(\frac{\Delta X(t) - E(\Delta X(t))}{\sqrt{V(\Delta X(t))}} \le \frac{-VaR(t) - E(\Delta X(t))}{\sqrt{V(\Delta X(t))}}\right) = \pi \quad (9)$$

We can thus deduce that:

$$\frac{-VaR(t) - E(\Delta X(t))}{\sqrt{V(\Delta X(t))}} = z_\pi \quad (10)$$

where z_π is the quantile of order π, value read from the table of the standard normal distribution. Several methods of calculating *VaR* assume that portfolio returns are normal and their distributions would be fully identified by their expectations and their standard deviations. However, it was shown that the assumption of normality of returns is often violated. Indeed, the form of the return distribution tends to deviate from the normal form and possesses what is called a leptokurtic form. It is thus advisable to correct the quantile of the Normal distribution.

The Cornish-Fisher expansion attenuates this insufficiency by approximating the relation between the quantile of a distribution and its moments. Indeed, it provides a more elaborated measure to correct the quantile of the distribution, based on the Taylor series. [9] provide this approximation by taking into account the third and fourth moments of a distribution.

The Cornish-Fisher expansion is written as follows:

$$w_\pi \cong z_\pi + \frac{1}{6}\left(z_\pi^2 - 1\right)S + \frac{1}{24}\left(z_\pi^3 - 3z_\pi\right)(K-3) - \frac{1}{36}\left(2z_\pi^3 - 5z_\pi\right)S^2 \quad (11)$$

With

S: The coefficient of skewness, characterizes the asymmetry of a distribution. It is associated at the third moment. Its formula is:

$$S = \frac{E\left[\Delta X_t - E(\Delta X_t)\right]^3}{\left(\sqrt{V(\Delta X_t)}\right)^3} \quad (12)$$

K: The coefficient of Kurtosis, characterizes the flatness of a distribution. It is associated at the fourth moment. It is used for distributions with fat tails. Its formula is:

$$K = \frac{E\left[\Delta X_t - E(\Delta X_t)\right]^4}{\left(\sqrt{V(\Delta X_t)}\right)^4} \quad (13)$$

To formulate the *VaR* equation, rewrite the dynamic as:

$$dX(t) = \frac{dS_t}{S_t} = \alpha_c dt + \sigma(t) dW_1(t) \quad (14)$$

And assuming

$$Z(t) = e^{-rt} X(t) \quad (15)$$

With

r is the discount rate

Applying *Itô* formula

$$dZ(t) = e^{-rt}\left(\alpha_c + X(t)\right)dt + e^{-rt}\sigma(t)dW_1(t) \quad (16)$$

The integration of this differential equation for the period $\Delta t = s - t$ gives:

$$Z(s) - Z(t) = \int_t^s e^{-r\tau}\left(\alpha_c + X(t)\right)d\tau + \int_t^s e^{-r\tau}\sigma(t)dW_1(\tau) \quad (17)$$

Using:

$$X(t) = e^{rt} Z(t) \quad (18)$$

We have

$$X(s) = e^{r(s-t)}X(t) + \frac{\alpha_c}{r}\left(e^{r(s-t)} - 1\right) + \int_t^s e^{r(s-\tau)}\sigma(t)dW_1(\tau) \quad (19)$$

Thus, the loss is defined as:

$$\Delta X(t) = X(s) - e^{r(s-t)}X(t) \quad (20)$$

Implying:

$$\Delta X(t) = \frac{\alpha_c}{r}\left(e^{r(s-t)} - 1\right) + \int_t^s e^{r(s-\tau)}\sigma(t)dW_1(\tau) \quad (21)$$

The first two moments of the loss are given by:

$$E[\Delta X(t)] = \frac{\alpha_c}{r}\left(e^{r(s-t)} - 1\right) \quad (22)$$

$$E[\Delta X(t)] = \frac{\sigma(t)^2}{-2r}\left(1 - e^{2r(s-t)}\right) \quad (23)$$

The approach to the calculation of VaR based on the Cornish-Fisher expansion aims to change the multiplier for the normal distribution in order to include third and fourth moments of the distribution. Then:

$$VaR(t) = \frac{\alpha_c}{r}\left(1 - e^{r\Delta t}\right) - w_\pi \sqrt{\frac{e^{2r\Delta t} - 1}{2r}\ \sigma(t)^2} \quad (24)$$

IV. Conclusion

The recent financial and economic crises reinforce the deficiency of Gaussian approach, which is why we have presented new model for *VaR* computation of non-Gaussian finance, which shows the interest in selecting the most adequate model to approach real situations. However, the new *VaR* approach should not only give more realistic results but also should be able to forecast the future *VaR* values.

We intend to continue this work by focusing on the *VaR* evaluation of models with jumps in the stochastic volatility such as the [10] model.

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