# Optimal Replenishment Policies, Credit Period and Shipments for an Integrated Supply Chain Model with Credit-linked Procurement Cost

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***Abstract*—In this paper, an integrated supply chain model with credit-linked procurement cost is established. The main contribution of this paper are as follows: (1) $n$ ($n$ is the number of shipments from supplier to retailer per production run, $T$ is the replenishment time interval of the retailer and $M$ is the credit period offered by the supplier), $T$ and $M$ all are decision variables. (2) Credit-linked procurement cost, i.e., the longer the credit period the higher the procurement cost. (3)The supplier grants the retailer upstream trade credit. Then, both cases of $T \geq M$ and $T < M$ are discussed thoroughly. Time horizon is considered to be infinite. Theoretical results are obtained and an easy-to-use algorithm is designed to investigate the optimal solutions. Finally, numerical examples are presented to illustrate the results. Sensitivity analysis with respect to critical parameters is also given and some managerial insights are gained.**

***Index Terms*—Supply chain management, Integrated model, Deterioration, Variable procurement cost, Trade credit**

## I. Introduction

Trade credit (delay in payments) arises when firms are capital constrained. Numerical examples and studies indicate that trade credit is a major short-term debt financing instrument. Goyal [1] first developed an economic order quantity cost model that the supplier allowed a fixed time period for the retailer to settle the account owed to him without charging any interest. Aggarwal and Jaggi [2] considered deteriorating items on the basis of Goyal’s model. Jamal, Sarker and Wang [3] established an EOQ model in which shortages and trade credit were taken into account. Huang [4] constructed a model incorporated up-stream and down-stream trade credit financing. Lately, Liao, Huang and Ting [5] developed an inventory model by considering two levels of trade credit, limited storage capacity. Wu, Al-khateeb, Teng and Cárdenas-Barrón [6] established a supplier-retailer-customer supply chain model. Shi et al. [7] discussed retailer’s optimal ordering policies for a single deteriorating item with ramp-type demand rate under permissible delay in payments.

All the above-mentioned literature take the length of credit period as an exogenous variable, differently, some researchers consider it as a decision variable. For example, Jaggi, Goyal and Goel [8] stated that credit period offered by the retailer to its customers has a positive impact on demand of an item. Teng and Lou [9], Wu, Ouyang, Cárdenas-Barrón and Goyal [10] taken the length of the credit period offered by the retailer as a decision variable. Recently, Chen and Teng [11] developed a supplier-retailer-customer supply chain model in which both of upstream and downstream are full trade credit financing, deterioration rate is assumed to be time-varying and discounted cash flow analysis is used to obtain the retailer's optimal credit period and cycle time.

Most of the aforesaid articles focused only on the supplier’s or buyer’s performance, in many circumstances each parties' local objectives may often conflict. Many scholars have noticed this and have did a lot of work on the integrated supply chain model. See Su, Ouyang, Ho and Chang [12], Chung, Liao, Ting, Lin and Srivastava [13], and Das, Das and Mondal [14].

Motivated by Das, Das and Mondal [15], the main contribution of our study are as follows. Firstly, some conditions in Das, Das and Mondal [15]) are relaxed. Such as, the whole business period is unnecessary assumed to be one year (i.e., $nT$ unnecessary equals to 1); the credit period offered by the supplier is unnecessary assumed to be within each replenishment period, so both cases of $T \geq M$ and $T < M$ are discussed thoroughly by using an alternative payment method. Secondly, in Das, Das and Mondal [15], the number of replenishment of the retailer $n$ is a constant, while in our study, the number of shipments from supplier to retailer per production run $n$, the replenishment time interval of the retailer $T$ and the credit period offered by the supplier $M$ all are decision variables. Then the optimal solutions are solved by an easy-to-use algorithm simultaneously. Thirdly, the existence and uniqueness of the optimal solutions are discussed completely. Theoretical results are obtained, and numerical examples are presented to illustrate the results. Sensitivity analysis with respect to major parameters is also given and some managerial insights are gained.

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## II. Notations and Assumptions

To develop the proposed model in this paper, we conclude the notations used in the following Table I.

TABLE I. NOTATIONS

| Symbol | Description |
|---|---|
| Parameters | |
| $A$ | The retailer's ordering cost per order. |
| $S$ | The supplier's setup cost per production run. |
| $V$ | The variable process cost to the supplier of dealing with the retailer. |
| $c$ | The retailer's procurement cost per unit item. |
| $u$ | The supplier's production cost per unit item. |
| $h_r$ | The retailer's inventory holding cost rate, excluding interest charge. |
| $h_s$ | The supplier's inventory holding cost. |
| $p_r$ | The retailer's selling price per unit item. |
| $D$ | The customer's demand rate per unit time. |
| $P$ | The supplier's production rate. |
| $\theta$ | The deterioration rate of the item for the retailer only. |
| $I_d$ | The interest rate of revenue deposited by retailer. |
| $I_c$ | The interest rate to be paid to the supplier for the remaining stock. |
| $I_s$ | The supplier's opportunity interest loss due to delay payment. |
| $Q$ | The initial quantity which is taken by the retailer for a cycle from the supplier. |
| $I(t)$ | The level of inventory at time $t\in[0,T]$ |
| Decision variables | |
| $T$ | The length of replenishment cycle time for the retailer. |
| $M$ | The retailer's trade credit period offered by the supplier. |
| $n$ | The shipments from supplier to retailer per production run. |

Assumptions:

(1) Next, the necessary assumptions to build the mathematical model are given below.

(2) The model involves a single supplier and a single retailer for a single product. With severe competition in modern market environment, we assume that shortages are not allowed to occur.

(3) The lead time is ignored and replenishment rate is instantaneous.

(4) Time horizon is infinite.

(5) Demand rate $D$ and production rate $P$ are assumed to be constant and $P$ is greater than $D$ to avoid shortages.

(6) In this model, we assumed that the deterioration is considered for retailer only at the rate of $\theta$, and supposed that the supplier is a big merchant who can have capacity to prevent deterioration. Besides, $\theta$ is supposed to be sufficient small.

(7) Retailer's procurement cost $c$ is assumed to be $c=c_0+c_1M$, where $c_0$ is the procurement cost in the absence of credit period and $c>0$.

(8) Supplier's process cost is defined as $V=V_0+V_1(Q-Q_0)$, for convenient discussion, $V_0$ is assumed to be a fixed process cost for the amount purchased by retailer which is equal to $Q_0$.

(9) During the trade credit period, the account is not settled and the sale revenue is deposited in an interest bearing account. Two scenarios are considered: $T\geq M$ and $T<M$. In the case of $T\geq M$, the retailer sells the goods and continues to earn the interest until $t=M$, then starts to pay the interest charged on the item in stock to the time $t=T$. In the case of $T<M$, the retailer doesn't need to pay any interest charged on the item in stock, Instead he/she can earn the interest from time $t=0$ to time $t=M$.

## III. The Integrated Supply Chain Model

The production and shipping policy are as follows: the supplier begins to product the item at time $t=0$ with the rate of $P$, and continuously to product the item until the total demand of the buyer is fulfilled, i.e., the supplier manufactures the item in batches of size $nQ$. Once the first $Q$ units are produced, the supplier delivers them to the retailer and then delivers the same amount of items $Q$ to the retailer after the time period $T$. This procedure up-to time $nT$ the moment that the supplier's inventory level falls to zero.

### A. The Inventory System for Retailer

The inventory system for retailer operates as follows. The replenishment at the beginning of the cycle brings the inventory level up to $Q$. Due to both demand and deterioration of the item, the inventory level gradually depletes during the period $[0,T]$ and falls to zero at $t=T$. Let $I(t)$ be the inventory level of deteriorating item at time $t$ for retailer. Then the differential equation of $I(t)$ for each replenishment cycle is:

$$\frac{dI(t)}{dt}+\theta I(t)=-D,\ 0\leq t\leq T. \tag{1}$$

With the boundary condition $I(0)=Q$ and $I(T)=0$. Solving (1) for the inventory over time, it yields

$$I(t)=\frac{D}{\theta}[e^{\theta(T-t)}-1]. \tag{2}$$

and the retailer's order size per cycle is

$$Q=I(0)=\frac{D}{\theta}[e^{\theta T}-1]. \tag{3}$$

For sufficient small $\theta$ (i.e., $\theta T\ll 1$), by Taylor expansion, $e^{\theta T}\approx 1+\theta T+\frac{(\theta T)^2}{2}$.

Hence, $Q\approx DT+\frac{1}{2}D\theta T^2$.

The retailer's average ordering cost ($ROC$) is $ROC=\frac{A}{T}$.

Retailer's average holding cost (*RHC*) (excluding interest charges) amounts to

$$RHC = \frac{ch_r}{T}\int_0^T I(t)dt = \frac{ch_r D}{\theta}(\frac{e^{\theta T}-1}{\theta T}-1) \approx \frac{ch_r DT}{2}.$$

Retailer's average deterioration cost (*RDC*) is given by

$$RDC = \frac{c}{T}(Q - \int_0^T Ddt) = cD(\frac{e^{\theta T}-1}{\theta T}-1) \approx \frac{cD\theta T}{2}.$$

For the interest earned and interest charged, from the values of the cycle time $T$ and credit period $M$, two different cases may arise: $T \ge M$ and $T < M$. According to Assumption (9), we will discuss the retailer's average interest earned and interest charged under these two cases respectively.

Let us discuss them separately.

**Case 1.** $T \ge M$.

Interest earned:

$$RIE_1 = \frac{p_r I_d}{T}\int_0^M \int_0^t Ddt = \frac{p_r I_d DM^2}{2T},$$

and interest charged:

$$RIC_1 = \frac{cI_c}{T}\int_M^T I(t)dt = \frac{cI_c D}{\theta}[\frac{e^{\theta(T-M)}-1}{\theta T} - \frac{T-M}{T}] \approx \frac{cI_c D(T-M)^2}{2T}.$$

**Case 2.** $T < M$.

Similar to Case 1, the interest earned and interest charged under this situation are as follows.

$$RIE_2 = \frac{p_r I_d}{T}[\int_0^T \int_0^t Ddt + DT(M-T)] = p_r I_d D(M - \frac{T}{2}),$$

and $RIC_2 = 0$.

From the above results, the retailer's average total cost $TR(T,M)$ is

$$TR(T,M) = \begin{cases} TR_1(T,M), & T \ge M \\ TR_2(T,M), & T < M. \end{cases}$$

Where

$$TR_1(T,M) = ROC + RHC + RDC + RIC_1 - RIE_1$$

$$= \frac{A}{T} + \frac{1}{2}DT(c_0 + c_1 M)(h_r + \theta) + \frac{I_c D(c_0 + c_1 M)}{2T}(T-M)^2 - \frac{p_r I_d DM^2}{2T}, \quad (4)$$

and

$$TR_2(T,M) = ROC + RHC + RDC + RIC_2 - RIE_2$$

$$= \frac{A}{T} + \frac{1}{2}DT(c_0 + c_1 M)(h_r + \theta) - p_r I_d D(M - \frac{T}{2}). \quad (5)$$

## B. The Inventory System for Supplier

The supplier starts to product goods at time $t = 0$. When $Q$ amount of item is produced, he/ she will deliver to the retailer as soon as possible and then will deliver the same amount item successively every time $T$ until $nT$ the time that the inventory reduced to zero. It is supposed that the supplier is a big merchant and manufactures the item in batches of size $nQ$. In light of the previous assumption, the supplier's average total relevant costs are given as the sum of average set up cost, average process cost, average inventory cost, average production cost and average opportunity interest loss cost. The concrete calculations are as follows.

Supplier's average set up cost (*SSC*) is $SSC = S/(nT)$.

Supplier's average process cost (SPV) is $SPV = V/T$. Supplier's average inventory cost (*SIC*) (details see Das, Das and Mondal [24])can be formulated as

$$SIC = \frac{h_s Q}{2}(n+1-\frac{nQ}{PT}) \approx \frac{h_s}{2}DT(1+\frac{1}{2}\theta T)(n+1-\frac{nD}{P}-\frac{nD\theta T}{2P}).$$

Supplier's average production cost (*SPC*) is

$$SPC = \frac{Ou}{T} \approx uD(1+\frac{1}{2}\theta T).$$

Supplier's average opportunity interest loss cost (*SOIL*) is

$$SOIL = \frac{I_s cMQ}{T} \approx I_s MD(c_0 + c_1 M)(1+\frac{1}{2}\theta T).$$

Therefore, supplier's average total cost $TS(n,T,M)$ is

$$TS(n,T,M) = SSC + SPV + SIC + SPC + SOIL$$

$$= \frac{S}{nT} + \frac{V_0 - Q_0 V_1}{T} + D[V_1 + u + I_s M(c_0 + c_1 M)](1+\frac{1}{2}\theta T) + \frac{h_s DT}{2}(n+1-\frac{nD}{P}-\frac{nD\theta T}{2P})(1+\frac{1}{2}\theta T). \quad (6)$$

## C. The Integrated Supply Chain Model for Deteriorating Item for Retailer And Supplier

As a result, in order to improve the competitiveness of the whole supply chain, the the supplier and retailer make an agreement to determine the best policy for the whole supply chain system. Under this circumstance, the average total cost $TC(n,T,M)$ of the integrated supply chain model for deteriorating item is given by:

$$TC(n,T,M) = \begin{cases} TC_1(n,T,M), & T \ge M \\ TC_2(n,T,M), & T < M. \end{cases}$$

where

$$TC_1(n,T,M) = TR_1(T,M) + TS(n,T,M)$$

$$= \frac{A}{T} + \frac{1}{2}DT(c_0 + c_1 M)(h_r + \theta) + \frac{I_c D(c_0 + c_1 M)}{2T}(T-M)^2 - \frac{p_r I_d DM^2}{2T} + \frac{S}{nT} + \frac{V_0 - Q_0 V_1}{T} + D[V_1 + u + I_s M(c_0 + c_1 M)](1+\frac{1}{2}\theta T) + \frac{h_s DT}{2}(n+1 - \frac{nD}{P} - \frac{nD\theta T}{2P})(1+\frac{1}{2}\theta T). \quad (7)$$

and

$$TC_2(n,T,M) = TR_2(T,M) + TS(n,T,M)$$

$$=\frac{A}{T}+\frac{1}{2}DT(c_0+c_1M)(h_r+\theta)-p_rI_dD(M-\frac{T}{2})+\frac{S}{nT}$$
$$+\frac{V_0-Q_0V_1}{T}+D[V_1+u+I_sM(c_0+c_1M)](1+\frac{1}{2}\theta T)$$
$$+\frac{h_sDT}{2}(n+1-\frac{nD}{P}-\frac{nD\theta T}{2P})(1+\frac{1}{2}\theta T). \quad (8)$$

It is obvious that $TC_1(n,M,M)=TC_2(n,M,M)$, so $TC(n,T,M)$ is a piecewise continuous function of $T$.

The objective is to determine the optimal length of replenishment cycle $T$ optimal trade credit period $M$ offered by the supplier and the optimal number of shipments $n$ from supplier to retailer per production run, such that $TC(n,T,M)$ is the minimum value.

## IV. The Theoretical Results

In order to determine the optimal solutions, we have the following theoretical results.

**Proposition 1.** For any given values of $T$ and $M$, $TC_i(n,T,M)(i=1,2)$ is a convex function on $n$.

**Proof.** Taking the first-order and second-order partial derivatives of $TC_i(n,T,M)(i=1,2)$ with respect to $n$, we obtain

$$\frac{\partial TC_1(n,T,M)}{\partial n}=\frac{\partial TC_2(n,T,M)}{\partial n}$$
$$=-\frac{S}{n^2T}+\frac{h_sDT}{2}(1-\frac{D}{P}-\frac{D\theta T}{2P})(1+\frac{1}{2}\theta T). \quad (9)$$
$$\frac{\partial^2 TC_1(n,T,M)}{\partial n^2}=\frac{\partial^2 TC_2(n,T,M)}{\partial n^2}=\frac{2S}{n^3T}. \quad (10)$$

It is clear that $\frac{\partial^2 TC_1(n,T,M)}{\partial n^2}=\frac{\partial^2 TC_2(n,T,M)}{\partial n^2}>0.$ Thus, for any given values of $T$ and $M$, $TC_i(n,T,M)(i=1,2)$ is a convex function on $n$. This completes the proof of Theorem 1.

Proposition 1 ensures that the search for the optimal shipment number $n$ is reduced to find a local optimal solution.

Further, the impact of other parameters on the number of shipments $n$ are given as the following Corollary 1.

**Corollary 1.** If $(D/P)\le(1/2)$, then it has

(1) A higher value of $h_s$ and $P$ cause a higher value of $n$.

(2) A higher value of $S$ causes a lower value of $n$.

**Proof.** From (9), we have

$$n=\sqrt{\frac{h_sDT^2}{2S}(1-\frac{D}{P}-\frac{D\theta T}{2P})(1+\frac{1}{2}\theta T)}. \quad (11)$$

Then the conclusion of Corollary 1 can be easy to get by (11). This completes the proof of Corollary 1.

Next, we will discuss the property of $TC_1(n,T,M)$

**Proposition 2.** (1) For any given values of $n$ and $M$, if $(D/P)\le(1/2)$, then $TC_1(n,T,M)$ is a strictly pseudo-convex function in $T$, and hence there exists a unique minimum solution $T_1^*$.

(2) If $(D/P)\le(1/2)$ and $\varphi(T,M)>0$, then there exists a unique solution $(n_1^*,T_1^*,M_1^*)$, such that $TC_1(n,T,M)$ is minimized. Where

$$\varphi(T,M)=2c_1I_cD(M-T)+(c_0+c_1M)I_cD-p_rI_dD+2c_1I_sD(1+\frac{1}{2}\theta T)T.$$

**Proof.** From (7), we define

$$G(n,T,M)=A+\frac{S}{n}+(V_0-Q_0V_1)+\frac{1}{2}DT^2(c_0+c_1M)(h_r+\theta)$$
$$-\frac{p_rI_dDM^2}{2}+\frac{I_cD(c_0+c_1M)}{2}(T-M)^2+D[V_1+u$$
$$+I_sM(c_0+c_1M)](1+\frac{1}{2}\theta T)T+\frac{h_sDT^2}{2}(n+1-\frac{nD}{P}$$
$$-\frac{nD\theta T}{2P})(1+\frac{1}{2}\theta T).$$

and

$H(T)=1/T>0$. Taking the first-order and second-order partial derivatives of $G(n,T,M)$ with respect to $T$, we obtain

$$\frac{\partial G(n,T,M)}{\partial T}=DT(c_0+c_1M)(h_r+\theta)+I_cD(c_0+c_1M)(T-M)$$
$$+D[V_1+u+I_sM(c_0+c_1M)](1+\theta T)$$
$$+\frac{3}{4}h_sD\theta T^2(n+1-\frac{2nD}{P}-\frac{2nD\theta T}{3P})$$
$$+h_sDT(n+1-\frac{nD}{P}). \quad (12)$$

and

$$\frac{\partial^2 G(n,T,M)}{\partial T^2}=D[(c_0+c_1M)(h_r+\theta+I_c+\theta I_sM)+\theta V_1$$
$$+\theta u+h_s(n+1-\frac{nD}{P})]+\frac{3}{2}h_sD\theta T(n+1$$
$$-\frac{2nD}{P}-\frac{nD\theta T}{P}). \quad (13)$$

Hence, if $(D/P)\le(1/2)$, then $(n+1-\frac{nD}{P})\ge\frac{1}{2}n>0$, and $(n+1-\frac{2nD}{P}-\frac{nD\theta T}{P})\ge(1-\frac{1}{2}n\theta T)$. For $\theta T\ll 1$, Hence, $(n+1-\frac{2nD}{P}-\frac{nD\theta T}{P})\ge(1-\frac{1}{2}n\theta T)>0$. So it can be deduced that $\frac{\partial^2 G(n,T,M)}{\partial T^2}>0$. Applying Theorems 3.2.9 and 3.2.10 from Cambini and Martein [16], we know that $TC_1(n,T,M)=G(n,T,M)/H(T)$ is a strictly pseudo-convex function in $T$. Hence, there exists a unique global minimum $T_1^*$. This completes the proof of Part (1) of Proposition 2.

$$\frac{\partial TC_1(n,T,M)}{\partial M}=\frac{1}{2T}[c_1D(h_r+\theta)T^2+c_1I_cD(T-M)^2$$
$$+2I_cD(c_0+c_1M)(M-T)-2p_rI_dDM$$
$$+2I_sD(c_0+2c_1M)(1+\frac{1}{2}\theta T)T]. \quad (14)$$

and

$$\frac{\partial^2 TC_1(n,T,M)}{\partial M^2}=\frac{1}{T}[2c_1I_cD(M-T)+I_cD(c_0+c_1M)$$
$$-p_rI_dD+2c_1I_sD(1+\frac{1}{2}\theta T)T]. \quad (15)$$

Combining (9) and (10), we have

$$\frac{\partial^2 TC_1(n,T,M)}{\partial n\partial M}=\frac{\partial^2 TC_1(n,T,M)}{\partial M\partial n}=0,$$

and $\frac{\partial^2 TC_1(n,T,M)}{\partial n^2}=\frac{2S}{n^3T}>0$. Taking $T$ as fixed, the determinant of the Hessian Matrix of $TC_1(n,T,M)$ with respect to $n$ and $M$ is obtained as follows.

$$|H_1|=\frac{\partial^2 TC_1(n,T,M)}{\partial n^2}\cdot\frac{\partial^2 TC_1(n,T,M)}{\partial M^2}=\frac{2S}{n^3T}\varphi(T,M).$$

Thus, if $\varphi(T,M)>0$, then $|H_1|>0$. In addition, combining the proof of part (1) of Proposition 2, we can conclude that, if $(D/P)\le(1/2)$ and $\varphi(T,M)>0$, then there exists a unique solution $(n_1^*,T_1^*,M_1^*)$, such that $TC_1(n,T,M)$ is minimized. This completes the proof of Part (2) of Proposition 2.

Furthermore, by setting $\frac{\partial TC_1(n,T,M)}{\partial n}=0$, $\frac{\partial TC_1(n,T,M)}{\partial T}=0$, and $\frac{\partial TC_1(n,T,M)}{\partial M}=0$, we can obtain the necessary and sufficient condition for the optimal solutions $n_1^*$, $T_1^*$, and $M_1^*$.

By using the analogous argument, we have the following Proposition 3.

**Proposition 2.** (1) For any given values of $n$ and $M$, if $(D/P)\le(1/2)$, then $TC_2(n,T,M)$ is a strictly pseudo-convex function in $T$, and hence there exists a unique minimum solution $T_2^*$.

(2) If $(D/P)\le(1/2)$, then there exists a unique solution $(n_2^*,T_2^*,M_2^*)$, such that $TC_2(n,T,M)$ is minimized.

**Proof.** The Proof of Proposition 3 is similar to the Proof of Proposition 2, thus we omit it.

Taking the first-order derivatives of $TC_2(n,T,M)$ with respect to $n$, $T$ and $M$ respectively, setting the results to zero, then the necessary and sufficient condition for the optimal solutions $n_2^*$, $T_2^*$, and $M_2^*$ can be obtained.

From $\frac{\partial TC_2(n,T,M)}{\partial M}=0$, we can derive the following Corollary 2.

**Corollary 2.** If $T<M$, then for any given value of $n$, $M(T)$ is decreasing on $T$. Where

$$M(T)=\frac{p_rI_d-\frac{1}{2}c_1T(h_r+\theta)-c_0I_s(1+\frac{1}{2}\theta T)}{2c_1I_s(1+\frac{1}{2}\theta T)}.$$

**Proof.** The Proof of Corollary 2 can be easily obtained by taking the first-order derivatives of $M(T)$ with respect to $T$, Hence, we also omit it.

Based on the above theoretical results, the optimal $n^*$, $T^*$ and $M^*$ can be got by the following algorithm.

**Algorithm.**

Step1. Input related parameters.

Step2. Start with $n=1$.

Step 3. Determine $T_1^{(n)}$ and $M_1^{(n)}$ by the equations of $\frac{\partial TC_1(n,T,M)}{\partial T}=0$ and $\frac{\partial TC_1(n,T,M)}{\partial M}=0$. If $T_1^{(n)}\ge M_1^{(n)}$, go to Step4; Otherwise, go to Step7.

Step 4. If $\varphi(T,M)>0$, $T_1^{(n)}\ge 0$ and $M_1^{(n)}\ge 0$, then calculate $TC_1(n,T_1^{(n)},M_1^{(n)})$. Otherwise, let $TC_1(n,T_1^{(n)},M_1^{(n)})=+\infty$.

Step 5. Set $n=n+1$, and repeat Step 3 and Step 4 until $T_1^{(n)}<M_1^{(n)}$.

Step 6. Find $TC_1^*(n_1^*,T_1^*,M_1^*)=\min\{TC_1(n,T_1^{(n)},M_1^{(n)})\}$, where $n$ are all these satisfied the condition $T_1^{(n)}\ge M_1^{(n)}$.

Step 7. Determine $T_2^{(n)}$ and $M_2^{(n)}$ by the equations of $\frac{\partial TC_2(n,T,M)}{\partial T}=0$ and $\frac{\partial TC_2(n,T,M)}{\partial M}=0$. If $T_2^{(n)}<M_2^{(n)}$, go to Step 8; Otherwise, there is no optimal solutions for the problem and stop.

Step 8. If $T_2^{(n)}\ge 0$ and $M_2^{(n)}\ge 0$, then calculate $TC_2(n,T_2^{(n)},M_2^{(n)})$. Otherwise, let $TC_2(n,T_2^{(n)},M_2^{(n)})=+\infty$.

Step 9. If $TC_2(n,T_2^{(n)},M_2^{(n)})\ge TC_2(n-1,T_2^{(n-1)},M_2^{(n-1)})$, then $TC_2^*(n_2^*,T_2^*,M_2^*)=TC_2(n-1,T_2^{(n-1)},M_2^{(n-1)})$, and the minimum value of all these satisfied the condition $T_2^{(n)}<M_2^{(n)}$ is obtained, then go to Step11 directly. Otherwise go to Step10.

Step 10. Set $n=n+1$, and repeat Step 7 to Step 9 until the minimum value of all these satisfied the condition $T_2^{(n)}<M_2^{(n)}$ is obtained.

Step 11. Let $TC^*(n^*,T^*,M^*)=\min\{TC_1^*(n_1^*,T_1^*,M_1^*),TC_2^*(n_2^*,T_2^*,M_2^*)\}$,

(1) if $TC^*(n^*,T^*,M^*)=TC_1^*(n_1^*,T_1^*,M_1^*)$, then $n^*=n_1^*$, $T^*=T_1^*$, and $M^*=M_1^*$ are the optimal solutions, and the corresponding values of $Q^*$, $c^*$, and $V^*$ can be determined subsequently.

(2) if $TC^*(n^*,T^*,M^*)=TC_2^*(n_2^*,T_2^*,M_2^*)$, then $n^*=n_2^*$, $T^*=T_2^*$, and $M^*=M_2^*$ are the optimal solutions, and the corresponding values of $Q^*$, $c^*$, and $V^*$ can be determined subsequently.

According to Corollary 2, once the procedure of the Algorithm goes into Step 7 it will not return to the previous Steps again.

## V. Numerical Examples And Sensitivity Analysis

To illustrate the model, this section applied the theoretical results and Algorithm to solve the following examples.

### A. *Examples to Illustrate the Solutions for the Integrated Model.*

**Example.** The input parameters are: $A=200$, $S=2000$, $V_0=50$, $V_1=0.5$, $Q_0=100$, $D=1000$, $P=2000$, $p_r=20$, $u=10$, $c_0=15$, $c_1=2.6$, $h_r=0.2$, $h_s=2$, $I_d=I_s=0.03$, $I_c=0.06$, $\theta=0.3$. Using the Algorithm to solve the problem and we get, $n^*=n_1^{(12)}=12$, $T^*=T_1^*=0.1725$, $M^*=M_1^*=0.1718$, $TC^*(n^*,T^*,M^*)=TC_1^*(n_1^*,T_1^*,M_1^*)=14805$, $Q^*=176.9353$, $c^*=15.4467$, and $V^*=88.4677$.

### B. *Sensitivity Analysis*

In this subsection, we mainly study the impact of the parameters $\theta$, $c_1$, and $V_1$ on the optimal solutions of the integrated model for deteriorating item.

The computational results shown in Table 1-3. Compared with Das, Das and Mondal [16], the mainly differences between Das, Das and Mondal [16] and ours are summarized as follows.

Table II shows the impact of $\theta$, $c_1$ and $V_1$ on decision variables and total cost per unit time, it can be seen that

(1) When $\theta$ increases from 0.10 to 0.28, $T^*<M^*$; while when $\theta$ increases from 0.34 to 0.46, $T^*\geq M^*$. Similarly, When $c_1$ increases from 0.60 to 2.40, $T^*<M^*$; while when $c_1$ increases from 3.00 to 4.20, $T^*\geq M^*$. When $V_1$ increases from 0.01 to 0.45, $T^*\geq M^*$; while when $V_1$ increases from 0.60 to 0.90, $T^*<M^*$. These results imply that our proposed model obtains the global optimal solutions for the integrated model in which the relationship between $T^*$ and $M^*$ contains $T^*\geq M^*$ and $T^*<M^*$. However, Das, Das and Mondal [16] considered only the case of $T^*\geq M^*$.

(2) It is obvious that $nT^*>1$ holds for all values of $\theta$, $c_1$ and $V_1$. It reveals that in today's fierce market competition, retailers and suppliers should consider all possible replenishment cycle time and number of shipments from supplier to retailer per production run to minimize the integrated model's average total cost. Hence, the whole business period unnecessarily equals to one year.

(3) It can be known that a larger $\theta$ results a smaller $T^*$ or $M^*$, while a larger $n^*$ or $TC^*$ ($TC^*(n^*,T^*,M^*)$). It means that when $\theta$ increases, retailers should take some actions to minimize their own average total cost as well as the integrated model's average total cost, for example, they can order more frequently and less quantity at a time, or they can also use some preservative measures to reduce deterioration rate or improve the equipment of storehouse.

(4) It can be found that as $c_1$ increases, $T^*$ and $TC^*$ increase, while $M^*$ decreases. Therein, $M^*$ is highly sensitive to $c_1$. Thus, in a sense, $c_1$ can be used to balance the benefits between suppliers and retailers.

(5) When $V_1$ increases, $T^*$ decreases, while $M^*$ and $TC^*$ increase. The results show that through the optimization of production process, reduce the waste of raw materials and improve the utilization ratio of the equipment, the supplier can reduce process cost effectively. Thus, the supplier can provide a longer credit period to his or her retailer, which can induce retailers order more quantity and less often. Meanwhile, retailers can make full use of the credit period, minimize their cost by order reasonably.

TABLE II. Impact of $\theta$, $c_1$ and $V_1$ on Decision Variables and Total Cost Per Unit TIME

| Parameters | | $T^*$ | $M^*$ | $n^*$ | $TC^*$ |
|---|---|---|---|---|---|
| $\theta$ | 0.10 | 0.2197 | 0.3766 | 9 | $TC_2^*=14297$ |
| | 0.16 | 0.1997 | 0.3112 | 10 | $TC_2^*=14463$ |
| | 0.22 | 0.1839 | 0.2544 | 11 | $TC_2^*=14617$ |
| | 0.28 | 0.1709 | 0.2039 | 12 | $TC_2^*=14760$ |
| | 0.34 | 0.1689 | 0.1648 | 12 | $TC_1^*=14893$ |
| | 0.40 | 0.1583 | 0.1523 | 13 | $TC_1^*=15019$ |
| | 0.46 | 0.1541 | 0.1438 | 13 | $TC_1^*=15139$ |
| $c_1$ | 0.60 | 0.1657 | 3.0891 | 12 | $TC_2^*=14632$ |
| | 1.20 | 0.1672 | 1.1997 | 12 | $TC_2^*=14754$ |
| | 1.80 | 0.1682 | 0.5686 | 12 | $TC_2^*=14789$ |
| | 2.40 | 0.1690 | 0.2517 | 12 | $TC_2^*=14802$ |
| | 3.00 | 0.1723 | 0.1614 | 12 | $TC_1^*=14808$ |
| | 3.60 | 0.1721 | 0.1465 | 12 | $TC_1^*=14812$ |
| | 4.20 | 0.1719 | 0.1320 | 12 | $TC_1^*=14816$ |
| $V_1$ | 0.01 | 0.1994 | 0.1855 | 10 | $TC_1^*=14572$ |
| | 0.15 | 0.1880 | 0.1801 | 11 | $TC_1^*=14641$ |
| | 0.30 | 0.1844 | 0.1783 | 11 | $TC_1^*=14713$ |
| | 0.45 | 0.1737 | 0.1725 | 12 | $TC_1^*=14783$ |
| | 0.60 | 0.1668 | 0.1895 | 12 | $TC_2^*=14847$ |
| | 0.75 | 0.1568 | 0.2346 | 13 | $TC_2^*=14910$ |
| | 0.90 | 0.1530 | 0.2519 | 13 | $TC_2^*=14967$ |

## VI. Conclusion

In this paper, we have complemented Das, Das and Mondal [16]'s model by: (1) relax some conditions their paper. Such as, the whole business period is unnecessary assumed to be one year (i.e., $nT$ unnecessary equals to $1$ ); the credit period offered by the supplier is unnecessary limited to be within each replenishment period, hence both cases of $T \geq M$ and $T < M$ are discussed thoroughly. (2) The number of shipments from supplier to retailer per production run $n$, the replenishment time interval of the retailer $T$ and the credit period offered by the supplier $M$ all are variables, and solved by an easy-to-use algorithm simultaneously. (3) The approximate solution of the integrated supply chain model is given by Taylor expansion, theoretical results have been obtained and numerical examples have provided, which show that: (a) The global optimal solution is indeed in the region of $T < M$, while Das, Das and Mondal [16] only considered the case of $T \geq M$. (b) To minimize the average total cost, the whole business period is not always equal to one year actually. If the retailer orders a lot of quantity, then the whole business period may exceed one year. Conversely, if the retailer orders small quantity, then the whole business may within one year. (c) Numerical examples and sensitivity analysis have confirmed the effective of the theoretical results and algorithm. In the meantime, we also highlight some managerial insights.

It is worthy to mention that this paper can continue to consider the coordinating mechanism to maintain long-term relationship between the supplier and the retailer. We will study the coordinating mechanism to distribute the saving cost by integrated supply chain model in the future research. This paper can also be explored along other directions. First, it can be extended to non-constant demand directly. Second, the factors of quantity discount, inflation can be added to it. Finally, the paper only considers a single supplier and a single retailer setting, a single supplier and multi-retailer setting is also worth discussing.

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